Number of degrees of freedom for a thermostat

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The dynamics of a quantum system which is weakly coupled with the quantum chaotic system with a few degrees of freedom is discussed. It is shown that the quantum chaotic system acts like the thermostat and causes the irreversible evolution of the system under interest. The closed equation of the system evolution is obtained. The solutions of this equation are compared with the results of the direct numerical simulation for two particular systems.

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I. INTRODUCTION

When we discuss the dynamics of a quantum system under the influence of the thermostat (environment) we imply that a thermostat possesses a number of properties which allow us to derive the closed equation of motion for the system under investigation. These properties are the following [1]: (i) the total density matrix factorizes in the direct product of the system density matrix and the density matrix of the thermostat; (ii) the correlation function of the thermostat decays with time. One justifies these properties by assuming the thermostat to have an infinite number of degrees of freedom and a continuous energy spectrum.

In this paper we show that these assumptions are surplus and, in fact, the thermostat can contain very few degrees of freedom—in the limit, only one. The only condition we have to satisfy is the condition of the chaotic dynamics of the "thermostat." It is shown in the paper that this property is enough to justify both assumptions (i) and (ii).

We begin the paper with a classical consideration of the problem. Today the fact that the classical chaotic system with a few degrees of freedom can act like a thermostat is well known. The study of this problem goes back to the paper of Fermi, Pasta, and Ulam [2] where they looked for the process of "self-thermalization" in the finite chain of the coupled nonlinear oscillators. It was shown later in [3] that thermalization in this system takes place if the condition of chaos is satisfied. The other example of the thermostat with a finite number of degrees of freedom is a multiatom molecule. We mention in particular the paper [4] where the absorption of the laser radiation by the multiatom molecule has been studied. It was shown that the finite number of the vibrational modes affects the resonant mode almost in the same way as it does the "ideal" thermostat with infinite number of freedom degrees. Again, the condition of chaos must be fulfilled. The similar problem of the energy absorption by the chaotic system is considered in [5]. Finally we should mention the recent paper [6]. In this paper the authors consider a two-dimensional chaotic map which models the Brownian motion. The completely deterministic derivation of the Fokker-Planck equation for the distribution function of the Brownian particle is presented and the exact expressions for the friction and diffusion coefficients are obtained.

In Sec. II of the present paper we analyze the outlined problem from the general viewpoint, irrelevant to any particular system. We obtain the closed equation of motion for the system weakly coupled with the chaotic system. Besides its own interest, this classical analysis helps us to "illuminate the weak links" which will cause the problems in the quantum consideration.

Our main interest is the case of the quantum thermostat with a finite number of degrees of freedom. The quantum analysis begins with the check of the properties (i) and (ii) for the quantum chaotic system [7]. This is done in Sec. III. As a particular model of the quantum chaotic system used in our numerical simulation, we have chosen the "quantum standard map on the torus" (SMT). Being coupled with some other system, this particular system is shown to effectively play the role of the thermostat. We illustrate this in Sec. IV for the two-level system and for the quantum kicked rotor. A comparison between the analytical solution and the direct numerical simulation of the composite system "system + thermostat" is given.

II. CLASSICAL ANALYSIS

Let us label the Hamiltonian H_a and the variables $q_a = (p_a, x_a)$ of the system under interest by the index a and by index b the Hamiltonian and the variables of the chaotic system b (we refer below to the system b as a thermostat). We additionally assume that interaction between the system and the thermostat is weak:

$$H_{tot} = H_a + H_b + \epsilon H_{int}, \quad H_{int} = V_a V_b, \quad H_b \text{ is chaotic} . \tag{1}$$

We want to obtain the closed equation for the system

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distribution function $\rho_a(q_a,t) = \int dq_b \rho_{tot}(q_a,q_b,t)$. In the analysis we will follow the corresponding quantum method well known in quantum mechanics [1].

Consider the arbitrary time t_0 and let

$$Q_\alpha = Q_\alpha(t,q_\alpha), \quad \alpha = a,b \tag{2}$$

be the solution of the Hamiltonian equations for $\epsilon = 0$ corresponding to the initial condition $Q_\alpha = q_\alpha$ for $t = t_0$. Using the substitution (2) we come to the distribution function in the "interaction picture"

$$\tilde{\rho}_{tot}(Q_a,Q_b,t) = \rho_{tot}(q_a(t,Q_a),q_b(t,Q_b),t) \tag{3}$$

$[q_\alpha = q_\alpha(t,Q_\alpha)$ is the inverse function to the function (2)], which satisfies

$$\partial\tilde{\rho}_{tot}(t)/\partial t = \epsilon\{\tilde{H}_{int}(t),\tilde{\rho}_{tot}(t)\} . \tag{4}$$

In Eq. (4) $\{\}$ is the Poisson bracket over the new variables Q_α and we write $\tilde{H}_{int}(t)$ to stress that in the new variables the interaction term depends on time even in the case when H_{int} does not contain the time explicitly. While $\epsilon \ll 1$ we can use the iteration method to solve Eq. (4). In second order over parameter ϵ we have

$$\tilde{\rho}_{tot}(t_0+\Delta t) = \tilde{\rho}_{tot}(t_0) + \epsilon\int_{t_0}^{t_0+\Delta t} dt\{\tilde{H}_{int}(t),\tilde{\rho}_{tot}(t_0)\} + \epsilon^2\int_{t_0}^{t_0+\Delta t} dt\int_{t_0}^{t} dt'\left\{\tilde{H}_{int}(t),\{\tilde{H}_{int}(t'),\tilde{\rho}_{tot}(t_0)\}\right\} . \tag{5}$$

Now we integrate both part of Eq. (5) over the variables of the thermostat. Having in mind that $\{\} = \{\}_a + \{\}_b$ and $\int dQ_b\{\}_b = 0$, we obtain, from Eq. (5),

$$\tilde{\rho}_a(t_0+\Delta t) = \tilde{\rho}_a(t_0) + \epsilon\int_{t_0}^{t_0+\Delta t} dt\{\tilde{V}_a(t),R_a^{(1)}(t)\} + \epsilon^2\int_{t_0}^{t_0+\Delta t} dt\int_{t_0}^{t} dt'\left\{\tilde{V}_a(t),\{\tilde{V}_a(t'),R_a^{(2)}(t,t')\}\right\} \tag{6}$$

where

$$R_a^{(1)}(t) = \int dq_b\tilde{V}_b(t)\tilde{\rho}_{tot}(t_0),$$
$$R_a^{(2)}(t,t') = \int dq_b\tilde{V}_b(t)\tilde{V}_b(t')\tilde{\rho}_{tot}(t_0) . \tag{7}$$

At this stage it is tempting to assume (as it is often done) that $\rho_{tot}(q_a,q_b,t) = \rho_a(q_a,t)\rho_b(q_b,t)$ for arbitrary t. But there are no reasons for this conjecture and in the general case it does not hold. We note, however, that for our purpose we do not need such an assumption. The problem can be settled by the theorem which follows from the mixing property of the chaotic dynamics.

Let $G(q_b)$ be an arbitrary function of the variables q_b, $H_{tot} = H_a + H_b$, and the dynamics of the system b is chaotic. Then

$$\lim_{t\to\infty}\int dq_b G(q_b)\rho_{tot}(q_a,q_b,t) = \langle G(q_b)\rangle_b\rho_a(q_a,t) , \tag{8}$$

where $\rho_a(q_a,t) = \int dq_b\rho_{tot}(q_a,q_b,t)$ and $\langle G(q_b)\rangle_b = \int dq_b G(q_b)\rho_b(q_b,t)$. In the case of the system with a confined chaotic region the average $\langle G(q_b)\rangle_b$ coincides with the average $\langle\langle G(q_b)\rangle\rangle$ over the invariant measure of the system b which coincides, in most cases, with the average over the chaotic part of the system's phase volume Γ:

$$\langle G(q_b)\rangle_b = \langle\langle G(q_b)\rangle\rangle \equiv \frac{1}{\Gamma}\oint_\Gamma dq_b G(q_b) . \tag{9}$$

We note also the consequence from Eq. (8):

$$\lim_{t\to\infty}\langle F(q_a)G(q_b)\rangle_{tot} = \langle F(q_a)\rangle_a\langle G(q_b)\rangle_b . \tag{10}$$

The numerical illustration of the properties (8) and (10) is given in Fig. 1(a) for the particular system with the Hamiltonian

$$H_{tot} = H_a + H_b, \quad H_\alpha = \frac{p_\alpha^2}{2} + K_\alpha\cos x_\alpha\sum_n\delta(t-n),$$
$$0 < x_\alpha, p_\alpha \le 2\pi \tag{11}$$

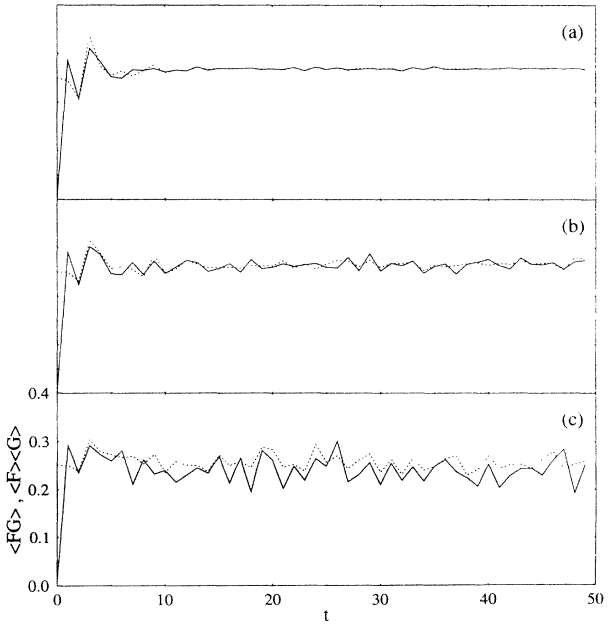

FIG. 1. The dynamics of the averages for SMT. The solid line shows $\langle\sin^2(p_a/2)\sin^2(p_b/2)\rangle_{tot}$ and the dashed line corresponds to $\langle\sin^2(p_a/2)\rangle_a\langle\sin^2(p_b/2)\rangle_b$. Parameters: $K_a = 0.5$, $K_b = 5$, $\epsilon = 0$, (a) $\hbar = 0$ (classical case), (b) $\hbar = 2\pi/512$, and (c) $\hbar = 2\pi/64$.

(the standard map on the torus). The values $K_b = 5$ and $K_a = 0.5$; thus the dynamics of the system b is chaotic while the dynamics of the system a is regular [8]. The solid line shows $\langle F(q_a)G(q_b)\rangle_{tot}$, where $F(q_a) = \sin^2(p_a/2)$ and $G(q_b) = \sin^2(p_b/2)$. The dashed line shows $\langle F(q_a)\rangle_a\langle G(q_b\rangle_b$. The initial distribution function was chosen in the form

$$\rho_{tot}(0) = \frac{1}{4\pi}[\delta(p_a - \pi)\delta(p_b) + \delta(p_a)\delta(p_b - \pi)] . \quad (12)$$

It is seen that, in spite of $\rho_{tot}(0)$ not allowing the factorization $[\rho_{tot}(0) \neq \rho_a(0)\rho_b(0)]$, the dashed line rapidly converges to the solid line, as predicted by Eq. (10).

We come back to Eq. (7). The theorem (8) states that for $t_0 \gg 1$

$$R_a^{(1)}(t) = \langle \tilde{V}_b(t)\rangle_b \tilde{\rho}_a(t_0),$$
$$R_a^{(2)}(t,t') = \langle \tilde{V}_b(t)\tilde{V}_b(t')\rangle_b \tilde{\rho}_a(t_0) . \quad (13)$$

Due to the chaotic dynamics of the system b the correlation function $S(t,t') = \langle \tilde{V}_b(t)\tilde{V}_b(t')\rangle_b$ rapidly decays

$$\langle \tilde{V}_b(t)\tilde{V}_b(t')\rangle_b \approx \langle \tilde{V}_b^2\rangle \exp(- \mid t - t' \mid /\tau_c) \quad (14)$$

and under the quite general condition $\langle \tilde{V}_b(t)\rangle_b = 0$. Now we assume that the characteristic frequency of the motion of the system a is much smaller than the inverse correlation time τ_c. It means that the function $\tilde{V}_a(t)$ in Eq. (6) is a slowly varying function on the time scale of τ_c. Using this assumption and taking $[\rho_a(t+\Delta t) - \rho_a(t)]/\Delta t \approx \partial\rho_a/\partial t$ we obtain the differential equation for the distribution function in the "interaction picture": $\partial\tilde{\rho}_a(t)/\partial t = \epsilon^2\langle \tilde{V}_b^2\rangle\tau_c\{\tilde{V}_a(t), \{\tilde{V}_a(t), \tilde{\rho}_a(t)\}\}$. Finally, coming back to the old variables q_a, we have the required closed equation for the system dynamics under the influence of the thermostat

$$\frac{\partial\rho_a(t)}{\partial t} = \{H_a, \rho_a(t)\} + D\{V_a(t), \{V_a(t), \rho_a(t)\}\},$$
$$D = \epsilon^2\langle V_b^2\rangle\tau_c . \quad (15)$$

Equation (15) is an irreversible equation. It means that the system a irreversibly loses information about its initial state. This information leaks into the chaotic system b which, due to the chaotic dynamics, has infinite "informational capacity." In this sense there is no difference between a chaotic system with one degree of freedom and the heat bath (the infinite ensemble of the linear oscillators); both possess infinite informational capacity. This justifies the use of the term "thermostat" for the system b.

One remark concerning the feedback is in order. As can easily be seen, the method used neglects the back action of the system on the thermostat. For our particular thermostat this approximation is well justified, because a small perturbation of the "standard map on the torus" with $K > 1$ changes neither its type of motion nor its equilibrium distribution function (invariant measure), which is $\rho_b(q_b) = 1/(2\pi)^2$. We note, however, that there are the chaotic thermostats which change considerably their invariant measure under an external perturbation. (The example of such a thermostat is considered in [6].) In this case Eq. (15) will contain an additional relaxation (friction) term in its right-hand side. In our case this term is negligibly small.

III. QUANTUM ANALYSIS

The consideration of the problem in the quantum case completely repeats all stages of the classical analysis. The only differences are that the distribution function is substituted by the density matrix and the Poisson bracket is substituted by the commutator. Therefore, all we have to do is to check the property of the factorization (8) and the property of the short memory (14) of the quantum chaotic system.

As a model we choose the quantum counterpart of the system (11)

$$\hat{H} = \hat{H}_a + \hat{H}_b + \epsilon\hat{H}_{int},$$
$$\hat{H}_\alpha = \frac{\hat{p}_\alpha^2}{2} + K_\alpha \cos x_\alpha \sum_n \delta(t - n) \quad (\alpha = a, b), \quad (16)$$
$$\hat{H}_{int} = \cos x_a \cos x_b \sum_n \delta(t - n), \quad 0 < x_\alpha \leq 2\pi .$$

The condition of the periodicity on momentum will be satisfied if we choose $\hbar = 2\pi/N$. Therefore, we consider dynamics of the two N-level quantum systems. The basis functions of every system obviously are

$$\mid p\rangle = (2\pi)^{-1/2}\exp(ipx), \quad x = 2\pi j/N, \quad 1 < j \leq N , \quad (17)$$

and $\hat{p} \mid p\rangle = \hbar p \mid p\rangle$, $1 \leq p \leq N$. For the purpose of the numerical simulation it is also convenient to choose N as a power of 2 [9].

A. Factorization in the quantum case

To avoid misunderstanding we remind the reader that by the "factorization" we mean the fulfillment of the conditions (8) and (10) but not the actual factorization of the total density matrix, which in the general case [excluding the case of the special initial condition $\hat{\rho}_{tot}(0) = \hat{\rho}_a(0) \otimes \hat{\rho}_b(0)$] never takes place.

Figures 1(b) and 1(c) illustrate the factorization property of the quantum chaotic system (16) for $N = 64$ and $N = 512$. The solid line shows $\langle \hat{F}\hat{G}\rangle_{tot}$, where $\hat{F} = \sin^2(\hat{p}_a/2)$, $\hat{G} = \sin^2(\hat{p}_b/2)$, and $\langle\rangle_{tot} = \mathrm{Tr}[\hat{\rho}_{tot}(t)\cdots]$. The dashed line shows $\langle \hat{F}\rangle_a\langle \hat{G}\rangle_b$, where $\langle\rangle_\alpha = \mathrm{Tr}[\hat{\rho}_\alpha\cdots]$ and $\hat{\rho}_\alpha = \mathrm{Tr}_{\beta\neq\alpha}[\hat{\rho}_{tot}(t)]$ are the reduced density matrices of the system a and b. The initial density matrix was chosen in a form which does not allow the factorization

$$\hat{\rho}_{tot}(0) = \frac{1}{2}(\mid 1\rangle\langle 1 \mid_a \otimes \mid N/2\rangle$$
$$\times\langle N/2 \mid_b + \mid N/2\rangle\langle N/2 \mid_a \otimes \mid 1\rangle\langle 1 \mid_b) . \quad (18)$$

It is seen that in the quantum case the factorization also takes place, but Eq. (10) holds up to finite accuracy.

The reason by which the factorization takes place in the quantum case is in the specific dynamics of the wave vector $\mid \psi_b(t)\rangle$ of a quantum chaotic system; for almost all initial conditions and almost all time moments one can consider the wave vector $\mid \psi_b(t)\rangle$ of the chaotic system as a random vector [10]. This quantum manifestation of the underlying classical chaos makes the following statement valid. Given the arbitrary operator $\hat{G}$ we have

$$\langle \psi_b(t) \mid \hat{G} \mid \psi_b'(t)\rangle = \frac{1}{N}\mathrm{Tr}(\hat{G}) + O(1/\sqrt{N}), \qquad (19)$$

when $\mid \psi_b(0)\rangle = \mid \psi_b'(0)\rangle$ and $\langle \psi_b(t) \mid \hat{G} \mid \psi_b'(t)\rangle = O(1/\sqrt{N})$ if $\mid \psi_b(0)\rangle \neq \mid \psi_b'(0)\rangle$. [It is easy to see that Eq. (19) is the quantum analoq of Eq. (9).] Using Eq. (19) we obtain that Eq. (7) in the quantum case is kept up to the accuracy $O(1/\sqrt{N}) \sim \sqrt{\hbar}$ and therefore the quantum results [Figs. 1(b) and 1(c)] converge to the classical one [Fig. 1(a)] approximately as $\sqrt{\hbar}$.

It is interesting to note that the accuracy up to which factorization holds in the quantum case greatly increases if the systems a and b are coupled and the coupling parameter satisfies the condition $\epsilon > \epsilon_{cr} \sim \hbar$ [see Figs. 2(b) and 2(c)]. This result shows the importance of the back action of the system on the thermostat in the quantum case. The physical reason for this phenomenon is the destruction of the quantum interference by an external stimulus. It has been shown in a recent paper [14] that the random perturbation converts the quantum dynamics of a semiclassical system into the classical dynamics, provided the intensity ϵ of the perturbation is more than $\epsilon_{cr} \sim \hbar$ [15]. In our case the feedback is definitely not random. In fact, it consists of the regular perturbation exerted by the system a itself (the system a is assumed to have regular dynamics for $\epsilon = 0$) and quasirandom perturbation originated by the thermostat b and returned back to it after some modification made by the system a. Nevertheless, as Figs. 2(b) and 2(c) show, even such a perturbation can force the quantum chaotic system to behave in a more classical way. We find similar results in [5] where the dynamics of a specific quantum system coupled with a finite number of the linear oscillators is studied.

To avoid any misunderstanding we stress that the effect discussed in the preceding paragraph is a pure quantum effect. In the classical case, as mentioned in Sec. III, the back action of the system on our thermostat is almost irrelevant [compare Figs. 1(a) and 2(a)].

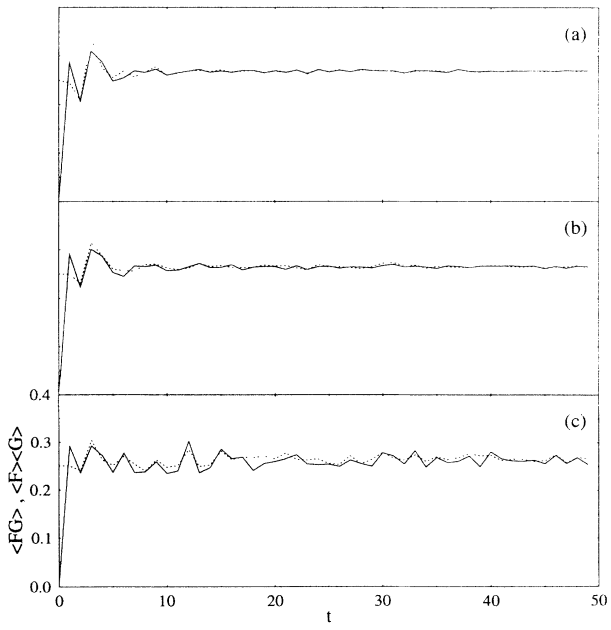

FIG. 2. The same as in Fig. 1, but $\epsilon \neq 0$: (a) $\epsilon = 1/64$, (b) $\epsilon = 1/64$, and (c) $\epsilon = 1/8$.

B. Correlation function of the quantum thermostat

In this subsection we discuss the "memory" property of the quantum chaotic system. The particular interest is the correlation function

$$S(\tau,t) = \mathrm{Tr}_b[\hat{U}_b^+(\tau)\hat{V}_b\hat{U}_b(\tau)\hat{V}_b\hat{\rho}_b(t)], \qquad (20)$$

$$\hat{U}_b(\tau) = \exp\left[-\frac{i}{\hbar}\int_t^{t+\tau}\hat{H}_b(t')dt'\right],$$

which corresponds to the classical correlator $\langle \tilde{V}_b(t+\tau)\tilde{V}_b(t)\rangle_b$ in Eq. (13). The behavior of $S(\tau,t)$ for $\tau = 1,2,9$ is shown in Fig. 3 for $\hbar = 2\pi/64$. The solid line corresponds to $\mid S(\tau,t) \mid$ and the dashed line corresponds to the imaginary part of $S(\tau,t)$. Figure 4 shows the func-

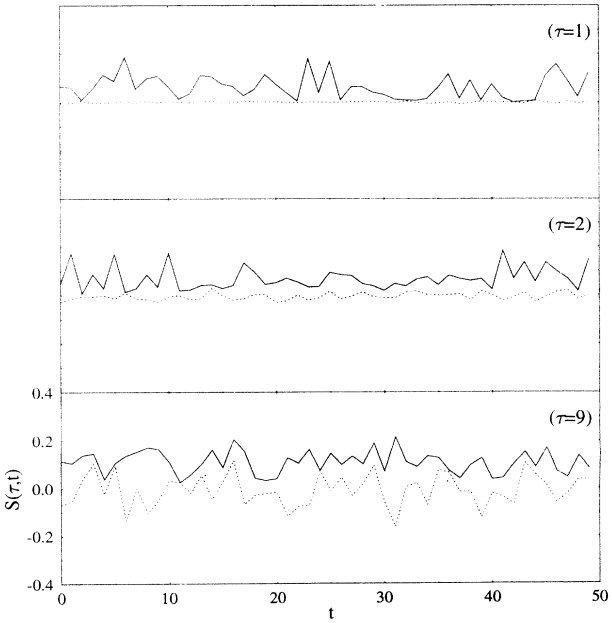

FIG. 3. The behavior of the correlation function $S(\tau,t)$ for quantum SMT ($K_b = 5$, $\hbar = 2\pi/64$). The solid line shows $\mid S(\tau,t) \mid$ and the dashed line shows the imaginary part of $S(\tau,t)$.

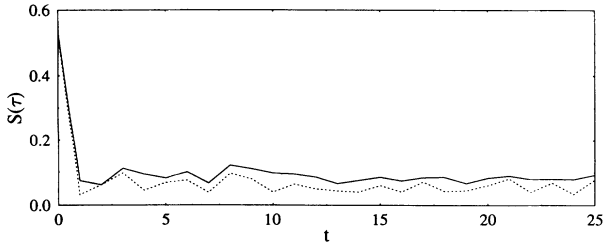

FIG. 4. The behavior of the correlation function $\bar{S}(\tau)$ (solid line). The parameters are the same as for Fig. 3. The average involves 25 kicks.

tion $\bar{S}(\tau) = \overline{|\, S(\tau,t)\,|}$, where the bar denotes the average over t. It is seen that the correlation function $\bar{S}(\tau)$ rapidly decays up to some finite value, which tends to zero if $\hbar \to 0$. For fixed $\hbar$ the level of the rest correlation can be reduced if there is the coupling between the systems, i.e., if we take into account the back action of the system on the thermostat (see dashed line in Fig. 4).

One can argue that the level of the rest correlation cannot always be small. Actually, from the mathematical viewpoint the dynamics of the correlation function is the quasiperiodic process with the frequencies given by the quasienergy frequencies. Therefore, there is some quasiperiod T in the behavior of $S(\tau,t)$. The question is how large is this quasiperiod? The simplest estimate shows that the quasiperiod grows exponentially with the number of the levels of the system b. In fact, for the considered system we have N values of the quasienergies and therefore approximately $N^2/2$ different quasienergy frequencies ω_i. Because of the underlying classical chaos these frequencies are randomly distributed over the whole frequency interval $0 < \omega_i < 2\pi$ [11]. We need to estimate the quasiperiod of a function of the type $f(t) = \sum_i a_i \cos(\omega_i t)$. Let us consider the most unfavorable case when all quasifrequencies are rational: $\omega_i = 2\pi r_i/q_i$; r_i, q_i are integer numbers $r_i < q_i$. Then the function $F(t)$ is periodic with the period $T = \prod_i q_i$. (The randomness of ω_i prevents the presence of the equal q_i.) While all $q_i > 1$ the period T grows exponentially with N: $T \sim \exp N \sim \exp(1/\hbar)$.

Thus, for small $\hbar$ (large N) the recurrence time T is very large and, in addition, one can increase it several times by taking into account the back action of the system. However, we cannot completely ignore the fact that there is a recurrence time for the quantum correlation function. (The finite recurrence time means the finite informational capacity of the quantum thermostat.) This fact causes the fundamental restriction on the time during which the closed equation for the system a [see Eq. (21) below] is valid.

C. Equation of the motion

The numerical results presented above show that the quantum chaotic system possesses to some extent both the factorization property and the property of short memory. These properties allow us to obtain the closed equation for the dynamics of a system weakly coupled with the chaotic quantum system. This equation obviously has the following form:

$$\frac{\partial \hat{\rho}_a(t)}{\partial t} = -\frac{i}{\hbar}[\hat{H}_a, \hat{\rho}_a] - \frac{D}{\hbar^2}[\hat{V}_a(t), [\hat{V}_a(t), \hat{\rho}_a(t)]],$$

$$D = \epsilon^2 \bar{S}(0)\tau_c \,. \quad (21)$$

Of course, Eq. (21) is approximate. It can be well justified only in the limit case, when the thermostat is classical. In other words, the number of levels N of the system b (more precisely, the number of level participating in the dynamics) should be infinite. Our aim is, however, the case of the finite N (finite $\hbar$). Given the above analytical estimates predicts Eq. (21) to hold with accuracy $\sim N^{-1/2}$ during the time $\sim \exp N$. We remind the reader that this estimate was obtained for the case $\epsilon \to 0$. For "large" ϵ ($\epsilon > \epsilon_{cr} \sim \hbar$) the feedback of the system makes the thermostat behave in a more classical way and the extent of the validity of Eq. (21) can be greatly enhanced. Preliminary estimates for this case gives the accuracy $\sim (NM)^{-1/2}$ during the time $\sim \exp(NM)$, where by M we denote the number of levels of the system a participating in the dynamics. In the next section we illustrate the extent of the validity of Eq. (21) for finite $\hbar$ for two particular systems $\hat{H}_a$.

IV. THE DYNAMICS OF A QUANTUM SYSTEM COUPLED WITH A CHAOTIC THERMOSTAT

A. Two-level system

We consider the simplest case of the Eq. (21) when the system a is a two-level system

$$\hat{H}_a = \hbar\omega \,|\, 2\rangle\langle 2\,|, \quad \hat{V}_a = |\, 2\rangle\langle 1\,| + |\, 1\rangle\langle 2\,| \,. \quad (22)$$

We choose the Hamiltonian of the composed system in the form

$$\hat{H}_{tot} = \hat{H}_a + \hat{H}_b + \epsilon \hat{V}_a \cos x_b \sum_n \delta(t-n),$$

$$\hat{H}_b = \frac{\hat{p}_b^2}{2} + K_b \cos x_b \sum_n \delta(t-n) \,. \quad (23)$$

We want to compare the case of the quantum thermostat with the case of the classical thermostat. In the latter case, as mentioned above, Eq. (21) is well justified and its solution has the form [12]

$$\rho_{1,2}(t) \approx \rho_{1,2}(0)\exp(-\gamma t - i\tilde{\omega}t),$$

$$\gamma = \epsilon^2/\hbar^2, \quad \tilde{\omega} = (\omega^2 + \gamma^2)^{1/2} \,. \quad (24)$$

Here we discuss the numerical methods. In the case of the classical thermostat we consider instead of Eq. (21) the following stochastic Schrödinger equation:

$$i\hbar\frac{\partial}{\partial t} \mid \psi_a(t)\rangle = \left[\hat{H}_a + \epsilon\hat{V}_a \cos x_b(t) \sum_n \delta(t-n)\right] \mid \psi_a(t)\rangle, \tag{25}$$

where $x_b(t)$ satisfies the classical equation of the motion. Then we obtain the density matrix of the system a by averaging the pure state $\mid \psi_a(t)\rangle\langle\psi_a(t) \mid$ over the initial distribution function of the classical thermostat. In the case of the quantum thermostat we solve the Schrödinger equation for the total wave function. Then we obtain the density matrix of the system a by tracing the total density matrix $\mid \psi_{tot}(t)\rangle\langle\psi_{tot}(t) \mid$ over the variables of the thermostat.

Figure 5(a) shows the dynamics of the real part of $\rho_{1,2}(t)$ in the case of the classical thermostat. The parameters are $\omega = 0.3$, $K_b = 5.0$, and $\epsilon/\hbar = 0.32$. The initial density matrix of the two-level system was chosen in the form $\hat{\rho}_a(0) = \frac{1}{2}(\mid 1\rangle\langle 1 \mid + \mid 2\rangle\langle 2 \mid + \mid 1\rangle\langle 2 \mid + \mid 2\rangle\langle 1 \mid)$ and the initial distribution function of the thermostat $\rho_b(p_b, x_b, 0) = \frac{1}{2\pi}\delta(p_b - \pi)$. It is seen that the behavior of $\hat{\rho}_a(t)$ obeys the formula (24) with $\gamma \approx 0.08$.

Figure 5(b) shows the case of the quantum thermostat ($\hbar = 2\pi/2048$) for two different initial $\hat{\rho}_{tot}(0)$:

$$\hat{\rho}_{tot}(0) = \frac{1}{2}(\mid 1\rangle\langle 1 \mid + \mid 2\rangle\langle 2 \mid + \mid 1\rangle \times\langle 2 \mid + \mid 2\rangle\langle 1 \mid)_a \otimes \mid N/2\rangle\langle N/2 \mid_b \tag{26}$$

(solid line) and

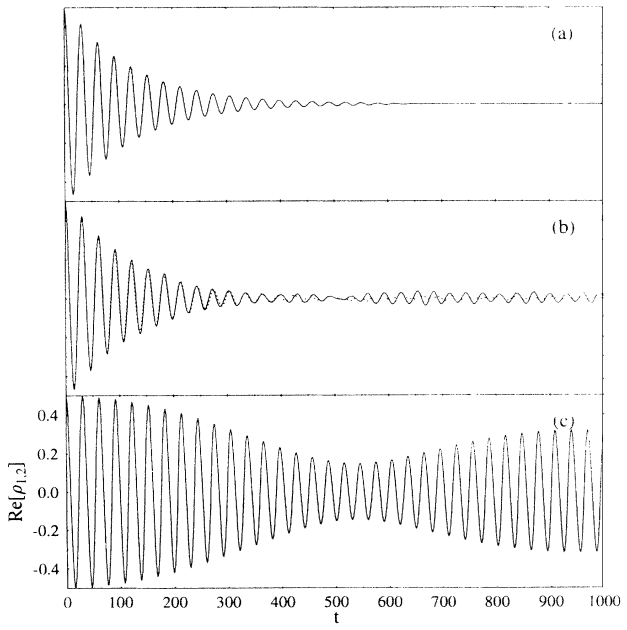

FIG. 5. The dynamics of the two-level system coupled with (a) the classical SMT and (b) and (c) the quantum SMT for $\hbar = 2\pi/2048$. Parameters: $\omega = 0.3$, (a) and (b) $K_b = 5$, and (c) $K_b = 0.5$ and $\epsilon/\hbar = 0.32$. Two curves in (b) correspond to the different choice of the initial condition: $\hat{\rho}_{tot} = \hat{\rho}_a \otimes \hat{\rho}_b$, solid line; $\hat{\rho}_{tot} \neq \hat{\rho}_a \otimes \hat{\rho}_b$, dashed line.

$$\hat{\rho}_{tot}(0) = \frac{1}{2}[(\mid 1\rangle\langle 1 \mid + \mid 2\rangle\langle 2 \mid + \mid 1\rangle\langle 2 \mid)_a \otimes \mid N/2\rangle \times\langle N/2 \mid_b + \mid 2\rangle\langle 1 \mid_a \otimes \mid 1\rangle\langle 1 \mid_b] \tag{27}$$

(dashed line). The similar behavior of both lines confirms again that in the case of the chaotic thermostat the requirement $\hat{\rho}_{tot} = \hat{\rho}_a \otimes \hat{\rho}_b$ is surplus. From a comparison of the cases (a) and (b) it is seen that the main difference between the classical and the quantum thermostat is the presence of the rest oscillation in the quantum case. We have found the level of the rest oscillation to be small during all times of the consideration, which was 8000 kicks.

To conclude this subsection we give numerical evidence that the condition of the chaotic dynamics of the thermostat is absolutely necessary. Figure 5(c) shows the case $K_b = 0.5$ when the dynamics of the system b is known to be regular. It is seen that there is no decay of $\rho_{1,2}(t)$ in this case.

B. Quantum kicked rotor

Now we proceed to the dynamics of the kicked rotor under the influence of the thermostat:

$$\hat{H}_a = \frac{\hat{p}_a^2}{2} + K_a \cos x_a \sum_n \delta(t-n),$$
$$\hat{H}_b = \frac{\kappa^2\hat{p}_b^2}{2} + K_b \cos x_b \sum_n \delta(t-n), \tag{28}$$
$$\hat{H}_{int} = \epsilon \cos x_a \cos x_b \sum_n \delta(t-n)\ .$$

In $\hat{H}_b$ we introduce the additional parameter κ, which allows us to change independently the semiclassical parameters of both systems, which are $\hbar$ and $\kappa\hbar$.

First we discuss the solution of Eq. (21) for $K_a > 1$. The solution of Eq. (21) for the kicked rotor strongly depends on the value of the parameter ϵ. For $\epsilon = 0$ the quantum kicked rotor exhibits the well known phenomenon of the suppression of the diffusion [13]. It means that for any finite $\hbar$ the mean energy of the rotor $E(t) = \frac{1}{2}\mathrm{Tr}[\hat{p}_a^2\hat{\rho}_a(t)]$ follows the classical behavior $E(t) \sim t$ only a finite time and then the energy growth is suppressed, as illustrated in Fig. 6 ($K_a = 2$ and $\hbar/2\pi = 21/2048$). For $\epsilon \neq 0$ the dynamics of the system becomes closer to the classical one and coincides with that if $\epsilon > \epsilon_{cr} \sim \hbar$ [9,14]. This case is depicted in Fig. 6 by the dashed line, which shows the result of the numerical simulation of the quantum kicked rotor under the influence of the classical thermostat.

The solid line in Fig. 6 shows the case of the quantum thermostat for $\kappa = 16/21$. For this value of κ the thermostat contains only $N = 128$ levels, but we see the excellent coincidence with the case of the classical thermostat. This coincidence is even better than it was for the two-level system where we took $N = 2048$. The reason for this is, of course, the back action of the system on the thermostat. If the back action is almost irrelevant in the case of the two-level system, it is very important

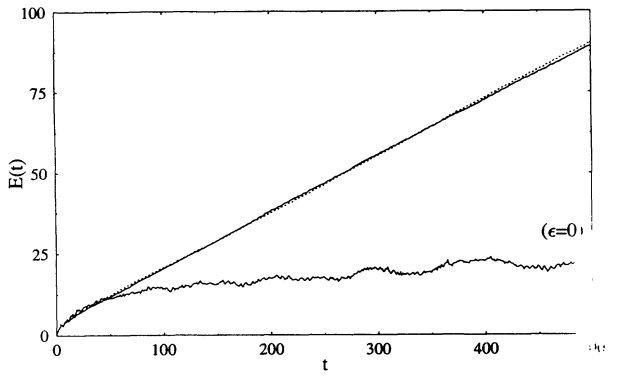

FIG. 6. The dynamics of the mean energy of the quantum kicked rotor coupled with the classical SMT (dashed line) and the quantum SMT (solid line). Parameters: $K_a = 2$, $K_b = 5$, $\hbar/2\pi = 21/2048$, $\epsilon = 1/16$, and $\kappa = 16/21$ in the case of the quantum SMT.

in the present case. It helps to validate the properties (i) and (ii) of the thermostat, which are crucial for validity of Eq. (21).

The result depicted in Fig. 6 is consistent with the numerical results reported in [9], where the dynamics of two coupled kicked rotors has been studied. It has been shown in [9] that the weak coupling between the rotors can greatly enhance the time of the correspondence between their quantum and classical dynamics [16]. Here we note that this phenomenon is not a particular feature of the kicked rotor, but has a general validity. As already mentioned, an arbitrary semiclassical system, whose dynamics obeys the irreversible Eq. (21), gets the transition to the classical dynamics if $\epsilon > \epsilon_{cr} \sim \hbar$ [14]. Therefore, in the case of the quantum thermostat (i.e., in the case studied in [9], where any of two rotors can be considered as the thermostat) the time of the correspondence is defined primarily by the validity time of Eq. (21). This time scales as

$$T \propto \exp(1/\hbar) \tag{29}$$

and it considerably exceeds the correspondence time of an isolated semiclassical system

$$T \propto 1/\hbar^{\alpha}, \quad \alpha > 0\,. \tag{30}$$

V. CONCLUSION

The numerical results presented show that the quantum chaotic system with a few degrees of freedom can act like the thermostat. Being coupled with some other quantum system it yields a "random" perturbation of the system and causes the irreversible evolution of the system according to Eq. (21).

A multiatom molecule might be the system where the discussed phenomenon is important. In particular, it might concern the problem of the suppression of the tunneling between the left and right isomers of the molecule when the mass of the molecule is increased. In fact, the more complex the molecule, the easier it satisfies the condition of chaos. Therefore, for the complex molecule the internal vibrational dynamics should be for certain chaotic. This chaotic motion influences the relevant tunneling degree of freedom (the double well system is usually used as the model) the way described above and causes the suppression of the tunneling, which is known to be a pure coherent effect.

The other physical application of the discussed phenomenon is the problem of the quantum-classical correspondence for the dynamics of the mesoscopic particle. This problem has been studied in [14] and the formulation of the problem considered there is very similar to that in the present paper: we have the Hamiltonian a, which is the Hamiltonian of the gravity center of the mesoscopic particle in some potential field $V(\mathbf{r})$, the Hamiltonian b, which is the Hamiltonian of the internal degrees of freedom of the particle, and the interaction between systems a and b, which is given by the term $H_{int} = \mathbf{d} \cdot \nabla V(\mathbf{r})$, where $\mathbf{d}$ is the dipole moment of the particle. It has been shown in [14] that under the assumptions (i) and (ii) about the internal motion one easily gets the correspondence between quantum and classical dynamics of the mesoscopic particle. In the cited paper these assumptions were justified by considering the particle's mass of the order of 0.1 g ($\sim 10^{22}$ internal degrees of freedom). The numerical results presented in Sec. IV show that there are the cases when we do not need such a huge number of the internal degrees of freedom to justify the assumptions (i) and (ii). Thus we can expect the quantum-classical correspondence even for a particle with few atoms.

ACKNOWLEDGMENTS

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